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A diffusion process for the asymptotic limit of a noncentred stochastic system

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Abstract

The diffusion effects of solution processes for a large class of stochastic equations have been characterized by Khasminskii's limit theory. Compared to either the Ito or the Stratonovich interpretation of stochastic differential equations, this theory has been effective from a modelling point of view in that the drift coefficient of the resultant Kolmogorov backward equation may include a term from the centred random field. A noncentred stochastic system on an asymptotically infinite interval is studied in this article on the basis of the limit theory and it is motivated by a singular behaviour of classical waves in a random multilayer. The extended Kolmogorov–Fokker–Planck equation for the transition probability density is derived and the solution of this equation is represented by an explicit approximate form based upon the pseudodifferential operator theory and Wiener's path integral representation.

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Many diffusion approximation problems for a system of differential equations with rapidly varying stochastic inputs are studied by replacing the equations by equations with a white-noise type of random idealization (Brownian motion) and interpreting the equations in the sense of Ito or Stratonovich. This leads to Fokker–Planck-type equations for the transition probability densities of the solution processes. From a modelling point of view, however, a limit theory covering a broader range of random fluctuations is required to apply the diffusion theory of stochastic differential equations to many practical problems. One generalization in this sense can be made by imposing a mixing condition on the underlying stochastic processes.

For a given mean zero random field $\boldsymbol{F}^\epsilon$, a large class of the stochastic processes solving the stochastic equations, not of Ito type,

$$(\mathrm{d}/\mathrm{d}\tau)\boldsymbol{x}^\epsilon(\tau,\sigma,\boldsymbol{x}) = \boldsymbol{F}^\epsilon(\tau/\epsilon, \boldsymbol{x}^\epsilon(\tau,\sigma,\boldsymbol{x})) \qquad \boldsymbol{x}^\epsilon(\sigma,\sigma,\boldsymbol{x}) = \boldsymbol{x} \in \mathrm{R}^d \tag{1}$$

(with a small parameter ϵ) converge weakly to a diffusion-type Markov process whose finite-dimensional distribution is determined by a parabolic partial differential equation with a certain infinitesimal generator, called *the Kolmogorov backward equation*. The point is that a nontrivial probability distribution of the solution process can be obtained over a growing time interval $O(\epsilon^{-1})$, which motivates the change of variables $\tau = t/\epsilon$. In terms of the new stretched variable, equation (1) becomes

$$(d/dt)\boldsymbol{x}^\epsilon(t,s,\boldsymbol{x}) = (1/\epsilon)\boldsymbol{F}^\epsilon(t/\epsilon^2, \boldsymbol{x}^\epsilon(t,s,\boldsymbol{x})) \qquad \boldsymbol{x}^\epsilon(s,s,\boldsymbol{x}) = \boldsymbol{x} \in \mathrm{R}^d \tag{2}$$

where $0 < s < t < t^*$ and t^* is $O(1)$. Note that, intuitively, if equation (2) is changed into an integral equation and ϵ in (2) is replaced by $1/\sqrt{n}$ (discretization), then the asymptotic behaviour of solution of (2) is likely to follow a type of central limit theorem. Motivated by this observation, we observe that if the random fluctuations in (2) are idealized by white noise, then (2) becomes a type of Ito equation; since the correlation length of the random fluctuations is $O(\epsilon^2)$, the white-noise idealization is achieved by replacing $(1/\epsilon)\boldsymbol{F}^\epsilon\, dt$ with $\boldsymbol{F}^*\, d\beta$, where $\boldsymbol{F}^*$ is a deterministic function and β is Brownian motion. Then the probability density for the solution process defined by the resultant Ito equation will satisfy the Fokker–Planck equation.

From a modelling point of view, however, Khasminskii's limit theorem [1] is more appropriate than either the Ito or the Stratonovich interpretation; the drift coefficient would include a term from the centred random field $\boldsymbol{F}^\epsilon$ in Khasminskii's theory. According to this theorem, under the assumption of a certain mixing condition—for example, an ergodic property—for the underlying process of $\boldsymbol{F}^\epsilon$, the finite-dimensional distribution represented by $E\{f(\boldsymbol{x}^\epsilon(t,s,\boldsymbol{x}))\}$, $f \in C^4(R^d)$, is approximated by $u^\epsilon(s,t,\boldsymbol{x};f)$ which solves the final-value problem

$$\partial_s u^\epsilon + \int_0^{1/\epsilon} E\{\boldsymbol{F}^\epsilon(s/\epsilon^2,\boldsymbol{x})\cdot\nabla(\boldsymbol{F}^\epsilon(s/\epsilon^2+t,\boldsymbol{x})\cdot\nabla u^\epsilon)\}\, dt = 0 \qquad u^\epsilon(t,t,\boldsymbol{x};f) = f(\boldsymbol{x}). \tag{3}$$

Here, the error of the approximation has been shown to be $O(\epsilon)$. The limit theory for such stochastic equations was first developed by Stratonovich [2] for problems of nonlinear vibrations in the presence of random noise. Then mathematical theory was developed by Khasminskii [1] and much of the fundamental extension has been done by a variety of authors. For example, Cogburn and Hersh's work in [3] allows a much broader class of stochastic perturbations and requires only a strong mixing condition. Papanicolaou and Kohler [4] extended the theory to include the linear problem.

This theory has become not only an important result in the study of diffusion processes [5], but also a powerful tool in many applications. These include the applications to the harmonic oscillator with randomly perturbed elastic constant, the diffusion approximation in transport theory, radio waves in turbulence, microwaves in a waveguide with a rough surface, and waves in geophysical media. In particular, both asymptotic and stochastic formulations of wave propagation problems in a type of complex medium use this type of theory to obtain successfully stochastic information on its signal and to probe the internal structure of the medium. References [6–8] give some examples of its applications.

The author considered in [7] the above system (2) over a further growing interval that is $O(\epsilon^{-1})$. The motivation for this comes from the consideration of a generalized case of wave propagation in which a singular behaviour of the waves may occur in a random medium due to the more general variations of the constitutive parameters. In this inner layer system, a stochastic system defined on the asymptotically infinite interval has to be studied for a uniformly valid limit theory.

Now, a nonzero deterministic field $\boldsymbol{G}^\epsilon$ is added to the system and this inclusion will be focused on. This term plays a role, for example, when a certain degree of wave dissipation is introduced. The generalized stochastic system to be considered in this article, therefore, is a noncentred system expressed by

$$\begin{aligned}&(\mathrm{d}/\mathrm{d}t)\boldsymbol{x}^\epsilon(t,s,\boldsymbol{x}) = (1/\epsilon)\boldsymbol{F}^\epsilon(t/\epsilon^2, \boldsymbol{x}^\epsilon(t,s,\boldsymbol{x})) + \boldsymbol{G}^\epsilon(t, \boldsymbol{x}^\epsilon(t,s,\boldsymbol{x}))\\ &\boldsymbol{x}^\epsilon(s,s,\boldsymbol{x}) = \boldsymbol{x} \in \mathrm{R}^d\end{aligned} \tag{4}$$

where $0 < s < t < t^{**}$ and t^{**} is $\mathrm{O}(\epsilon^{-1})$.

The underlying process of stochastic inputs in (4) is assumed to be a mixing process satisfying the requirement that $\rho(t)$, which is defined as the supremum of $|P(A|B) - P(B)|$ over $A \in F_{s+t}^{\infty}$, $B \in F_0^s$ where $s \geqslant 0$, vanishes as t goes to ∞. Here, F_s^t are nondecreasing σ-algebras. If, given this strong mixing condition, the limit theorem in [7] can be extended to (4), then the finite-dimensional distribution of the solution process of (4) will be approximated by the solution of the corresponding Kolmogorov backward equation with a certain infinitesimal generator. The transition probability density of the process then solves, as a forward variable, the adjoint equation of the Kolmogorov backward equation, and its explicit representation will be obtained by the pseudodifferential operator theory combined with an infinite-dimensional functional construction (Wiener's path integral).

The problem is that of whether the limit theorem still holds on the $\mathrm{O}(\epsilon^{-1})$ interval with the extra deterministic field $\boldsymbol{G}^\epsilon$. The two results in [7] and [9] can imply the required extension. One can obtain the result (without additional difficulties) by combining the limit theorem in [7] for $(\mathrm{d}/\mathrm{d}t)\boldsymbol{x}^\epsilon = (1/\epsilon)\boldsymbol{F}^\epsilon$ on the $\mathrm{O}(1/\epsilon)$ interval and the limit theorem in [9] for $(\mathrm{d}/\mathrm{d}t)\boldsymbol{x}^\epsilon = (1/\epsilon)\boldsymbol{F}^\epsilon + (1/\epsilon)\boldsymbol{G}^\epsilon$ on the $\mathrm{O}(1)$ interval.

To express the extended version of the final-value problem (3), one needs to define some notation. First, let $\overline{\boldsymbol{x}}_s^t(\boldsymbol{x}) = \overline{\boldsymbol{x}}^\epsilon(t,s,\boldsymbol{x})$ denote the solution of the deterministic problem $(\mathrm{d}/\mathrm{d}t)\overline{\boldsymbol{x}}^\epsilon(t,s,\boldsymbol{x}) = \boldsymbol{G}^\epsilon(t, \overline{\boldsymbol{x}}^\epsilon(t,s,\boldsymbol{x}))$, in which the stochastic term of (4) is suppressed, with the initial condition $\overline{\boldsymbol{x}}^\epsilon(s,s,\boldsymbol{x}) = \boldsymbol{x} \in R^d$. Also, in terms of the first and second derivatives of $\overline{\boldsymbol{x}}^\epsilon(t,s,\boldsymbol{x})$, denoted by $\boldsymbol{D}_s^t(\boldsymbol{x}) = \nabla \otimes \overline{\boldsymbol{x}}_s^t$ and $\boldsymbol{S}_s^t(\boldsymbol{x}) = (\nabla \otimes \nabla) \otimes \overline{\boldsymbol{x}}_s^t$, respectively, one defines tensor functions $\boldsymbol{Q}(s,t,\boldsymbol{x}) = (\boldsymbol{D}_t^s(\overline{\boldsymbol{x}}_s^t) \otimes \boldsymbol{D}_t^s(\overline{\boldsymbol{x}}_s^t)) \cdot \boldsymbol{D}_s^t(\boldsymbol{x})$, $\boldsymbol{R}_1(s,t,\boldsymbol{x}) = \boldsymbol{D}_t^s(\overline{\boldsymbol{x}}_s^t) \otimes \boldsymbol{D}_s^t(\overline{\boldsymbol{x}})$, and $\boldsymbol{R}_2(s,t,\boldsymbol{x}) = \boldsymbol{S}_t^s(\overline{\boldsymbol{x}}_s^t) \cdot \boldsymbol{D}_s^t(\overline{\boldsymbol{x}})$. Then the extended version of the Kolmogorov backward equation is given by

$$\begin{aligned}\partial_s u^\epsilon + \boldsymbol{G}^\epsilon \cdot \nabla u^\epsilon + \int_s^{s+\epsilon} &\mathrm{d}t\, \epsilon^{-2} E\{(\boldsymbol{F}^\epsilon(s, s/\epsilon^2, \boldsymbol{x}) \otimes (\nabla \otimes \boldsymbol{F}^\epsilon(s, t/\epsilon^2, \overline{\boldsymbol{x}}_s^t))) : \boldsymbol{R}_1(s,t,\boldsymbol{x})\\ &+ (\boldsymbol{F}^\epsilon(s, s/\epsilon^2, \boldsymbol{x}) \otimes \boldsymbol{F}^\epsilon(s, t/\epsilon^2, \overline{\boldsymbol{x}}_s^t)) : \boldsymbol{R}_2(s,t,\boldsymbol{x})\} \cdot \nabla u^\epsilon\\ &+ \int_s^{s+\epsilon} \mathrm{d}t\, \epsilon^{-2} E\{(\boldsymbol{F}^\epsilon(s, s/\epsilon^2, \boldsymbol{x})\\ &\otimes \boldsymbol{F}^\epsilon(s, t/\epsilon^2, \overline{\boldsymbol{x}}_s^t)) : \boldsymbol{Q}(s,t,\boldsymbol{x})\} : (\nabla \otimes \nabla u^\epsilon) = 0\end{aligned} \tag{5}$$

where ':' denotes the inner product of two tensor fields.

There are three components of the drift vector and one component of the diffusion matrix in (5). These are denoted by $\boldsymbol{G}^\epsilon$, $\boldsymbol{R}_1^\epsilon$, $\boldsymbol{R}_2^\epsilon$, and $\boldsymbol{Q}^\epsilon$, sequentially. One can observe that the first term, $\boldsymbol{G}^\epsilon$, of the drift vector corresponds to the one that appears in the diffusion limit theory of the Ito stochastic equations. This term plus the second one, $\boldsymbol{G}^\epsilon + \boldsymbol{R}_1^\epsilon$, is a modified term in the known Khasminskii's limit theory of centred mixing stochastic equations. Now, for the noncentred mixing stochastic equations, one more drift term, $\boldsymbol{R}_2^\epsilon$, is now added to this drift term. This third term would appear only if $\boldsymbol{G}^\epsilon$ does not vanish. For if this deterministic field vanishes, then $\boldsymbol{S}_s^t$ becomes the zero tensor for all s and t and so is $\boldsymbol{R}_2^\epsilon$. Therefore, the Kolmogorov backward equation (5) is an extension of (3). Also, note that the infinitesimal generator of (5) demonstrates how the solution of the effective system $(\mathrm{d}/\mathrm{d}t)\overline{\boldsymbol{x}}^\epsilon(t,s,\boldsymbol{x}) = \boldsymbol{G}^\epsilon(t, \overline{\boldsymbol{x}}^\epsilon(t,s,\boldsymbol{x}))$ couples with the random field $\boldsymbol{F}^\epsilon$ of stochastic system (4).

If the result (5) is combined with the principle of averaging in [10], then it follows immediately that $E\{f(\boldsymbol{x}^\epsilon(t,s,\boldsymbol{x}))\}$ converges uniformly in s and t to $u(s,t,\boldsymbol{x};f)$ which solves the equation $(\partial_s + \overline{L}_s)u(s,t,\boldsymbol{x};f) = 0$ with the final condition $\lim_{s\uparrow t} u(s,t,\boldsymbol{x};f) =$

$f(\boldsymbol{x}) \in C^4\ (R^d)$, where $\overline{L}_s$ is the limit of the infinitesimal generator of equation (5) as ϵ goes to zero.

Now, the asymptotic Kolmogorov backward equation given by (5) is changed into an asymptotic forward equation for the transition probability density function $P^\epsilon(s, \boldsymbol{x}; t, \boldsymbol{y})$. As known from Ito's theory, one can show similarly from Dynkin's formula that this satisfies $\partial_t P^\epsilon + \nabla \cdot ((\boldsymbol{G}^\epsilon + \boldsymbol{R}_1^\epsilon + \boldsymbol{R}_2^\epsilon) P^\epsilon) - (\nabla \otimes \nabla) : (\boldsymbol{Q}^\epsilon P^\epsilon) = 0$, called the *Kolmogorov–Fokker–Planck equation*, with the initial condition $\lim_{t \downarrow s} P^\epsilon(s, \boldsymbol{x}; t, \boldsymbol{y}) = \delta(\boldsymbol{y} - \boldsymbol{x})$. If perturbation analysis is applied to this equation, then the leading-order transition probability density $P^0(s, \boldsymbol{x}; t, \boldsymbol{y})$ will satisfy

$$\partial_t P^0 + \lim_{\epsilon \to 0} \int_0^{1/\epsilon} \mathrm{d}s\, E\{\nabla \cdot (\boldsymbol{G}^\epsilon(t, \boldsymbol{y}) + \boldsymbol{F}^\epsilon(t, t/\epsilon^2, \boldsymbol{y}) \cdot (\nabla \otimes \boldsymbol{F}^\epsilon(t, t/\epsilon^2 + s, \boldsymbol{y})))\} P^0$$
$$- \lim_{\epsilon \to 0} \int_0^{1/\epsilon} \mathrm{d}s\, E\{(\nabla \otimes \nabla) : (\boldsymbol{F}^\epsilon(t, t/\epsilon^2, \boldsymbol{y}) \otimes \boldsymbol{F}^\epsilon(t, t/\epsilon^2 + s, \boldsymbol{y}))\} P^0 = 0 \qquad (6)$$

with the initial condition $\lim_{t \downarrow s} P^0(s, \boldsymbol{x}; t, \boldsymbol{y}) = \delta(\boldsymbol{y} - \boldsymbol{x})$. Since the limit of $\boldsymbol{R}_2^\epsilon(t, \boldsymbol{y})$ as ϵ goes to zero is zero for all t and $\boldsymbol{y}$, only the lower-order terms of P^ϵ will be affected by $\boldsymbol{R}_2^\epsilon(t, \boldsymbol{y})$.

In general, it is difficult to express the solution representation of equation (6) in closed form due to the dependence of the coefficients on the variables t and $\boldsymbol{y}$. The pseudodifferential operator theory is applied here and it is combined with Wiener's path integral representation to approximate solutions of (6). References [11] and [12] can be referred to for the relevant general theory. In this case, the corresponding operator symbol contains the complete spectral information for the transition probability density.

To obtain the explicit approximate form of the solution of (6), let the interval $[s, t]$ be divided into a number N of subintervals such that $s = t_0 < t_1 < \cdots < t_N = t$ with the corresponding values $\boldsymbol{y}_i$ evaluated at t_i. Let $L^*(t, \boldsymbol{y})$ denote the adjoint operator of the infinitesimal generator (6). In terms of $\boldsymbol{G}^0$, $\boldsymbol{R}_1^0$, $\boldsymbol{R}_2^0$, and $\boldsymbol{Q}^0$ as the limit of $\boldsymbol{G}^\epsilon$, $\boldsymbol{R}_1^\epsilon$, $\boldsymbol{R}_2^\epsilon$, and $\boldsymbol{Q}^\epsilon$ as ϵ goes to zero, it is given by

$$L^*(t, \boldsymbol{y}) = \boldsymbol{Q}^0 \cdot (\nabla \otimes \nabla) - (\boldsymbol{G}^0 + \boldsymbol{R}_1^0 - 2\nabla \cdot \boldsymbol{Q}^0) \cdot \nabla + \nabla \cdot (\boldsymbol{G}^0 + \boldsymbol{R}_1^0) - (\nabla \otimes \nabla) \cdot \boldsymbol{Q}^0. \quad (7)$$

In terms of the operator symbol $\Lambda_{L^*}(t; \boldsymbol{y}, \boldsymbol{p})$ belonging to the symbol class $S^2_{1,0}$ corresponding to the operator L^*, one can use the pseudodifferential operator theory to recast equation (6) in the form

$$\partial_t P^0 + (2\pi)^{-2} \int_{R^4} \mathrm{d}\boldsymbol{y}'\, \mathrm{d}\boldsymbol{p}\, \mathrm{e}^{\mathrm{i}\boldsymbol{p} \cdot (\boldsymbol{y} - \boldsymbol{y}')} \Lambda_{L^*}(t; \boldsymbol{y}', \boldsymbol{p}) P^0 = 0. \qquad (8)$$

The solution representation for (8) can be directly expressed in terms of path integrals. To account for the t-dependence in the operator symbol, one can use repeatedly the well known Chapman–Kolmogorov equation on each subinterval $[t_{j-1}, t_j]$ to take the transition probability density function as a time-ordered product. In conjunction with the pseudodifferential operator analysis of [12], the solution of equation (8) then takes the approximate form given by

$$P^0(s, \boldsymbol{x}; t, \boldsymbol{y}) = \lim_{N \to \infty} (2\pi)^{-N} \int_{R^{2(2N-1)}} \mathrm{d}\boldsymbol{y}_1\, \mathrm{d}\boldsymbol{y}_2 \cdots \mathrm{d}\boldsymbol{y}_{N-1}\, \mathrm{d}\boldsymbol{p}_1\, \mathrm{d}\boldsymbol{p}_2 \cdots \mathrm{d}\boldsymbol{p}_N$$
$$\times \exp\left\{ \mathrm{i} \sum_{j=1}^{N} (\boldsymbol{p}_j \cdot (\boldsymbol{y}_j - \boldsymbol{y}_{j-1}) + ((t-s)/N) \Lambda_{L^*}(t_j; \boldsymbol{y}_j, \boldsymbol{p}_j)) \right\}. \qquad (9)$$

The results of this paper are now summarized. Motivated by the asymptotic and stochastic formulation of wave propagation problems in random media, an analysis of a noncentred stochastic system on an asymptotically infinite interval has been provided. It shows that the inclusion of a deterministic term in Khasminskii's theorem in an asymptotically infinite interval still admits the diffusion approximation. The principle of averaging and perturbation analysis are used to obtain the Kolmogorov–Fokker–Planck equation for the leading-order probability density function. From the pseudodifferential operator theory and Wiener's path integral theory, finally, this equation is recast as an integrodifferential equation and the solution is approximated by an infinite-dimensional functional form.

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